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Further Exact Solutions of the Eight-Vertex SOS Model and Generalizations of the Rogers–Ramanujan Identities

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The restricted eight-vertex solid-on-solid (SOS) model is an exactly solvable class of two-dimensional lattice models. To each site i of the lattice there is associated an integer height l_i restricted to the range $1 \leqslant l_i \leqslant r-1$. The Boltzmann weights of the model are expressed in terms of elliptic functions of period $2K$, and involve a parameter η. In an earlier paper we considered the case $\eta = K/r$. Here we generalize those considerations to the case $\eta = sK/r$, s an integer relatively prime to r. We are again led to generalizations of the Rogers–Ramanujan identities.

KEY WORDS: Statistical mechanics; lattice statistics; number theory; eight-vertex model; solid-on-solid model; hard-hexagon model; Rogers–Ramanujan identities.

1. THE LOCAL HEIGHT PROBABILITIES P_a

1.1. Introduction

In a previous paper to be referred to as A,[1] the restricted eight-vertex solid-on-solid (SOS) model was solved when the parameter η had the value

$$\eta = K/r \tag{1.1.1}$$

r denoting a positive integer and $2K$ the period of the elliptic functions which naturally occur. When $r=5$ the model is equivalent to a previously solved[2] hard square model, which includes the hard hexagon model. When $r=4$ it has been noted by Huse[3] that the model is equivalent to the zero field Ising model.

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Using the corner transfer matrix (C.T.M.) technique the local height probabilities P_a were calculated for the large but finite system. Then a generalization of I. Schur's[4] proof of the Rogers–Ramanujan identities was used to transform the expressions obtained from the C.T.M.s into a from suitable for taking the thermodynamic limit.

In this paper we repeat the considerations of A for

$$\eta = sK/r \tag{1.1.2}$$

where s and r are relatively prime integers (r positive). In general the model will no longer be physical as there will be negative Boltzmann factors. Nevertheless, the calculation of the height "probabilities" again involve generalizations of identities of the Rogers–Ramanujan type. The model is therefore of independent interest for this feature.

1.2. The Restricted SOS Model with $\eta = sK/r$

The restricted SOS model has been defined in A. Consider a square lattice $\mathscr{L}$. With each site i associate an integer height l_i. Impose the condition that heights on adjacent sites must differ by 1. There are six possible configurations, as shown in Fig. 1. Furthermore restrict the heights

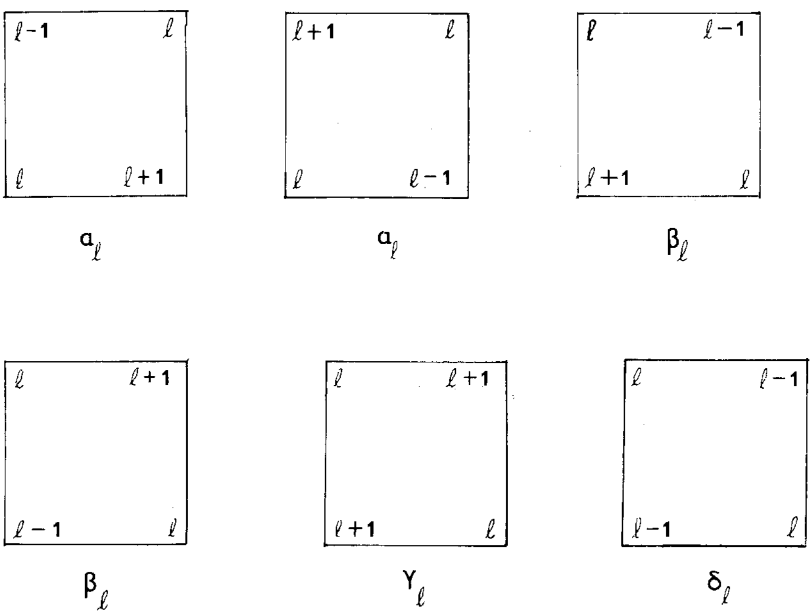


Fig. 1. The six possible arrangements of heights round a face of the lattice.

to lie in the interval $1 \leqslant l_i \leqslant r-1$. With the sites of a face ordered as in Fig. 2, associate a weight $W(l_i, l_j \mid l_m, l_n)$. We take for the weights [Eqs. (1.2.12) of A]

$$\begin{gathered} W(l, l+1 \mid l-1, l) = W(l, l-1 \mid l+1, l) = \alpha_l \\ W(l+1, l \mid l-1) = W(l-1, l \mid l, l+1) = \beta_l \\ W(l+1, l \mid l, l+1) = \gamma_l, \qquad W(l-1, l \mid l, l-1) = \delta_l \end{gathered} \tag{1.2.1}$$

where

$$\begin{aligned} \alpha_l &= \rho' h(v+\eta) \\ \beta_l &= \rho' h(\eta - v)[h(w_{l-1})\, h(w_{l+1})]^{1/2}/h(w_l) \\ \gamma_l &= \rho' h(2\eta)\, h(w_l + \eta - v)/h(w_l) \\ \delta_l &= \rho' h(2\eta)\, h(w_l - \eta + v)/h(w_l) \end{aligned} \tag{1.2.2}$$

The elliptic theta function h is defined by

$$h(v) = 2p^{1/4} \sin \frac{\pi v}{2K} \prod_{n=1}^{\infty} \left(1 - 2p^n \cos \frac{\pi v}{K} + p^{2n}\right)(1-p^{2n})^2 \tag{1.2.3}$$

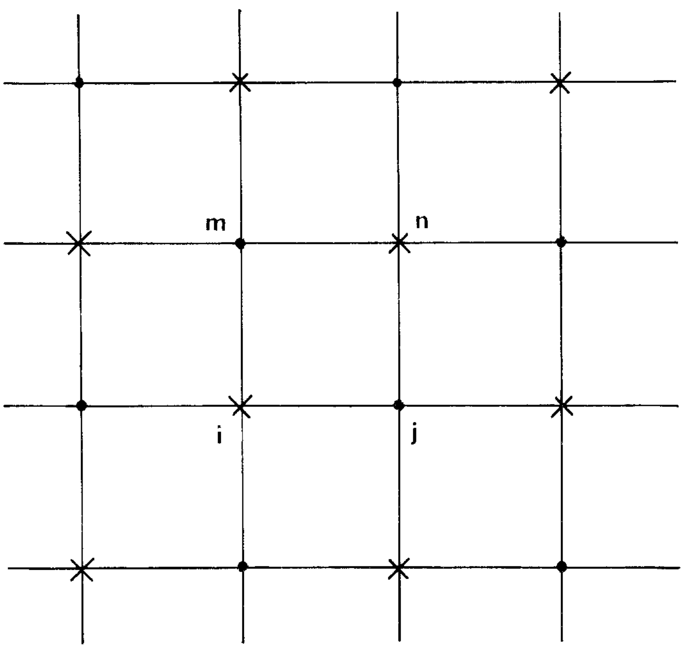


Fig. 2. The square lattice $\mathscr{L}$, showing a typical face (i, j, n, m) and the two sublattices X and Y (denoted, respectively, by dots and crosses).

where K and K' are the complete elliptic integrals of the first kind, and p is the nome

$$p = e^{-\pi K'/K} \tag{1.2.4}$$

which we take to lie in the interval $-1 < p < 1$. The quantity ρ' is an arbitrary constant, and w_l is given by

$$w_l = 2l\eta \tag{1.2.5}$$

We are considering the case when η is given by (1.1.2). We regard p and η as parameters, and v as a complex variable.

The partition function is

$$Z = \sum \prod W(l_i, l_j \mid l_m, l_m) \tag{1.2.6}$$

where the sum is over all allowed height configurations, and the product is over all faces of the lattice. The local height probability is

$$P_a = Z^{-1} \sum \delta(l_1, a) \prod W(l_i, l_j \mid l_m, l_n) \tag{1.2.7}$$

where the δ is the Kronecker delta, and the product has the same meaning as in (1.2.6). l_1 is the height of the center site 1 of the lattice, a is an integer between 1 and $r-1$. Clearly we have

$$\sum_{a=1}^{r-1} P_a = 1 \tag{1.2.8}$$

It was shown in A that if the model defined by the weights (1.2.2) with the restriction that neighbouring heights differ by 1 has the property

$$h(w_0) = h(w_r) = 0 \tag{1.2.9}$$

then the star triangle relation[5] is satisfied. With w_l given by (1.2.5) and η given by (1.1.2) the property (1.2.9) holds so the model again satisfies the star triangle relation.

1.3. Symmetry Properties of the Partition Function

It was shown in A that the partition function Z and the P_a, assuming toroidal boundary conditions, are unchanged if $W(l, m' \mid l', m)$ is multiplied by

$$\omega^{l-m} g_l g_m / g_{l'} g_{m'} \tag{1.3.1}$$

where $\omega^4 = 1$ and g_l is arbitrary. For example, the choice

$$\omega = i, \qquad g_l = 1 \tag{1.3.2}$$

negates β_l and leaves the other weights unchanged.

The elliptic theta function has the properties

$$h(v) = -h(-v) = -h(v + 2K) \tag{1.3.3}$$

Using (1.3.3) we see either of the substitutions

$$v \to v + 2K \qquad \text{or} \qquad \eta \to \eta + 2K \tag{1.3.4}$$

merely negate each of the weights and thus leave Z and P_a unchanged. The substitution

$$v \to -v \tag{1.3.5}$$

and then making the transformation (1.3.1) with $\omega = 1$, $g_l = [1/h(w_l)]^{1/2}$ interchanges the weights α_l and β_l, and replaces γ_l by δ_{l+1} and δ_l by γ_{l-1}. From Fig. 1 this corresonds to a rotation of the lattice through 90°, and thus leaves Z and P_a unchanged. Similarly the substitution

$$\eta \to -\eta \tag{1.3.6}$$

corresponds to a rotation of the lattice. If we make both the substitutions

$$v \to K - v \qquad \text{and} \qquad \eta \to K - \eta \tag{1.3.7}$$

and then make a transformation of the form (1.3.2) to negative β_l, both Z and P_a are unchanged.

The elliptic theta function has the quasiperiodicity property

$$h(v + iK') = (p)^{-1/2} e^{-\pi i v/K} h(v) \tag{1.3.8}$$

Thus the substitution

$$v \to v + iK' \tag{1.3.9}$$

and then making the transformation (1.3.1) with $\omega = 1$ and $g_l = \exp(-\pi i \eta l^2/2K)$ multiplies each weight by the constant

$$c = (p)^{-1/2} e^{-\pi i v/K} \tag{1.3.10}$$

Hence Z is multiplied by c^N (N = number of faces of the lattice), and the P_a are unchanged.

When $0 < p < 1$ the symmetry properties (1.3.4), (1.3.6), and (1.3.7) imply it suffices to consider the region $0 < \eta < K$ and $-\eta < \mathrm{Re}(v) < 2K - \eta$. Recalling $\eta = sK/r$, it is convenient to break the region into "regimes." Labeling them in an analogous manner to those in A we define the following:

$$\begin{aligned}
&\text{Regime II:} && 0 < p < 1, && s = 1, \eta < \mathrm{Re}(v) < K - \eta \\
&: && 0 < p < 1, && s = 2, 3,..., r - 1, \eta < \mathrm{Re}(v) < 2K - \eta \\
&\text{Regime III:} && 0 < p < 1, && s = 1, 2,..., r - 2, -\eta < \mathrm{Re}(v) < \eta \\
&: && 0 < p < 1, && s = r - 1, K - \eta < \mathrm{Re}(v) < \eta \\
&\text{Regime VIII:} && 0 < p < 1, && s = r - 1, -(K - \eta) < \mathrm{Re}(v) < K - \eta
\end{aligned} \qquad (1.3.11a)$$

Regime VIII has been reported for hard hexagons in Ref. 6. We have not included the region $s = 1$, $K - \eta < \mathrm{Re}(v) < K + \eta$, the region $s = 1$, $K + \eta < 2K - \eta$ nor the region $s = r - 1$, $-\eta < \mathrm{Re}(v) < \eta - K$. From the symmetry (1.3.7) the first of these regions is identical to Regime VIII, while from the symmetries (1.3.7) and (1.3.6) the other two regions are identical to Regime II with $s = 1$.

When $-1 < p < 0$ and thus $\mathrm{Im}(K') = K$ the additional symmetry (1.3.8) implies it suffices to consider the region $0 < \eta < K/2$, $-\eta < \mathrm{Re}(v) < K - \eta$. Breaking the region into regimes, we define

$$\begin{aligned}
&\text{Regime I:} && -1 < p < 0, && s = 1, 2,..., [(r - 1)/2], \eta < \mathrm{Re}(v) < K - \eta \\
&\text{Regime IV:} && -1 < p < 0, && s = 1, 2,..., [(r - 2)/2], -\eta < \mathrm{Re}(v) < \eta \\
&: && -1 < p < 0, && r \text{ odd}, s = (r - 1)/2, (K/2) - \eta < \mathrm{Re}(v) < \eta \\
&\text{Regime X:} && -1 < p < 0, && r \text{ odd}, s = (r - 1)/2, \\
& && && -((K/2) - \eta) < \mathrm{Re}(v) < (K/2) - \eta
\end{aligned} \qquad (1.3.11b)$$

where [ ] denotes the integer part. Here, for r odd and $s = (r - 1)/2$ we have not included the region $-\eta < \mathrm{Re}(v) < (K/2) - \eta$ since from the symmetry (1.3.6) this region is identical to Regime IV with $s = (r - 1)/2$.

1.4. The Weights in Terms of the Conjugate Modulus

The above regimes were defined so that their boundary separates regions with different ground states. To investigate the ground states it is necessary to convert the weights to conjugate modulus form. This has been done in A [Eq. (A37) if $0 < p < 1$, (A58), if $-1 < p < 0$]. However the expressions in A can be simplified if we use the transformation (1.3.1) to remove the factors involving the g_l from all the weights, and the factors $w^{1/2}$, $w^{-1/2}$ in the weights (A58). Defining the function $E(z, x)$ for all complex z and $|x| < 1$ by

$$E(z,x) = \prod_{n=1}^{\infty} (1 - x^{n-1}z)(1 - x^n z^{-1})(1 - x^n)$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n x^{n(n-1)/2} z^n \tag{1.4.1}$$

we then have

$$\begin{aligned} \alpha_l &= v w^{1/2} E(xw^{-1}) \\ \beta_l &= v(xE_{l-1}E^{l+1}/(wE_l^2))^{1/2} E(w) \\ \gamma_l &= vE(x)\,E(x^l w)/E(x^l) \\ \delta_l &= vE(x)\,E(x^l w^{-1})/E(x^l) \end{aligned} \tag{1.4.2}$$

where v is independent of x, w, and l and thus irrelevant [it is given explicitly in A, Eqs. (A35) and (A56) if we divide those expressions by $E(x)$],

$$E_l = E(x^l, y) \tag{1.4.3}$$

and we have written $E(z, y)$ simply as $E(z)$. The quantities x, y and w are dependent on the sign of p. Denote

$$x = e^{-4\pi\eta/M'}, \qquad w = e^{-2\pi(\eta - v)/M'} \tag{1.4.4}$$

Then if $0 < p < 1$,

$$M' = K', \qquad y = x^{r/s} \tag{1.4.5}$$

while if $-1 < p < 0$ (and thus $K' = L' + iK$)

$$M' = 2L', \qquad y = -x^{r/2s} \tag{1.4.6}$$

From the conjugate modulus form of the weights we can determine whether or not the model is physical, or can be made physical by a transformation of the type (1.3.1) (by physical we mean that all the Boltzmann weights are positive). Using the product expansion of the E function (1.4.1) we conclude the only physical cases are Regime III $s = r - 1$ ($y^{1/r} < w < 1$), $s = 1$; Regime II $s = 1$ ($1 < w < x^{-1}$), $s = r - 1$; Regime I $s = 1$ ($1 < w < x^{-1}$); Regime IV $s = 1$.

Let us consider the model defined by (1.4.2) and (1.4.4). Define y by the relation

$$y^{\mu}=x^{r}, \qquad 1\leqslant\mu\leqslant r-1 \tag{1.4.7}$$

We can then give a unified treatment of all the regimes defined by (1.3.11) if we allow x to take both positive and negative values with

$$|x|<1 \text{ and } y^{t}\neq x^{l} \text{ for any } t \text{ and } l=1,2,\ldots,r-1 \tag{1.4.8}$$

[This is essential for the weights (1.4.2) to be well defined]. To see this first not that analogous to the symmetry property (1.3.7) we have P_a unchanged by the substitutions

$$\text{both } x\to y/x \qquad \text{and} \qquad w\to 1/w \tag{1.4.9}$$

Define a generalized Regime III (III_G, say) by

$$\text{Regime } \mathrm{III}_G: \quad |x|<|w|<1, \quad |y|<|x|<1 \tag{1.4.10}$$

By writing the definitions of Regimes III, IV, VIII, X (1.3.11) in terms of x, y, and w we see immediately these regimes are contained in Regime III_G. Furthermore the mapping (1.4.9) takes both Regimes I and II into Regime III_G. Hence denoting the height probability in Regime III_G by $P_a^{\mathrm{III}_G}(x,y,\mu)$ (in choosing this notation we have used the fact, to be established subsequently, that the P_a are independent of w), and in Regime T ($T=\mathrm{I},\ldots,X$) by $P_a^T(r,s,y)$ we have

$$\begin{aligned}
P_a^{\mathrm{I}}(r,s,y)&=P_a^{\mathrm{III}_G}(-e^{-2\pi(K/2-\eta)/L'},-e^{-\pi K/L'},r-2s)\\
P_a^{\mathrm{II}}(r,s,y)&=P_a^{\mathrm{III}_G}(e^{-4\pi(K-\eta)/K'},e^{-4\pi K/K'},r-s)\\
P_a^{\mathrm{III}}(r,s,y)&=P_a^{\mathrm{III}_G}(e^{-4\pi\eta/K'},e^{-4\pi K/K'},s)\\
P_a^{\mathrm{IV}}(r,s,y)&=P_a^{\mathrm{III}_G}(e^{-2\pi\eta/L'},-e^{-\pi K/L'},2s)\\
P_a^{\mathrm{VIII}}(r,r-1,y)&=P_a^{\mathrm{III}_G}(e^{-4\pi(K-\eta)/K'},e^{-4\pi K/K'},r-1)\\
P_a^{\mathrm{X}}(r,(r-1)/2,y)&=P_a^{\mathrm{III}_G}(e^{-2\pi\eta/L'},-e^{-\pi K/L'},r-1)
\end{aligned} \tag{1.4.11}$$

Note in particular

$$P_a^{\mathrm{III}}(r,s,y)=P_a^{\mathrm{II}}(r,r-s,y) \tag{1.4.12}$$

which is a consequence of the symmetry (1.3.7).

1.5. The Ground States

We have seen that Regimes I–X are contained in Regime III_G. Hence in discussing the ground states it suffices to discuss this generalized regime, with μ restricted to the values given by (1.4.11).

First note that Regime III_G is not distinct, in the sense that the P_a calculated in the region

$$|x| \leqslant |w| \leqslant |x|^{1/2}, \qquad |y| < |x| < 1 \tag{1.5.1}$$

can be deduced from the P_a calculated in the region

$$|x|^{1/2} \leqslant |w| \leqslant 1, \qquad |y| < |x| < 1 \tag{1.5.2}$$

This follows from the mapping [analogous to (1.3.6)]

$$w \to x/w \tag{1.5.3}$$

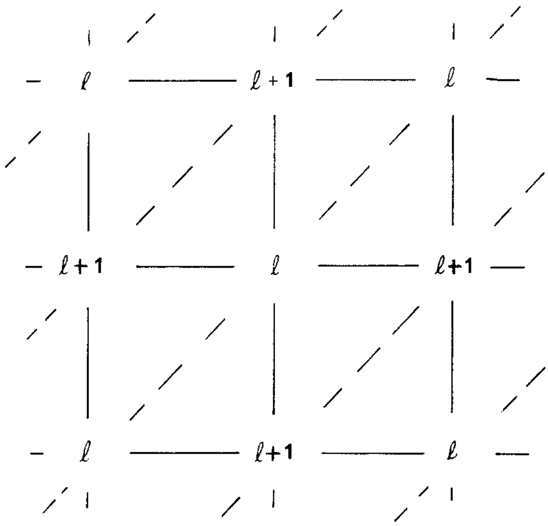
which rotates the lattice through 90° and thus leaves the P_a unchanged. Hence it suffices to consider the region (1.5.2).

We seek the ground states configuration(s) of the partition function defined by the weights (1.4.2). The ground state configuration is defined as that which maximizes the absolute value of the partition function (allowing for multiplicities) in the limit $M' \to 0$, which corresponds to extreme order or disorder. From (1.4.4) and (1.4.7) we see this is equivalent to taking $x, y, w \to 0$, provided $|w| \neq 1$ i.e., $\mathrm{Re}(v) \neq \eta$, a condition we shall assume. We consider four candidates, which are analogues of known ground states of the hard hexagon model and the Ising model (see Fig. 3). Clearly other ground states can be constructed by translations of those given in Fig. 3. Thus there are two possibilities for each N_l and S_l (height l on the X or Y sublattice), four possibilities for each Q_l, and $2r-4$ for R.

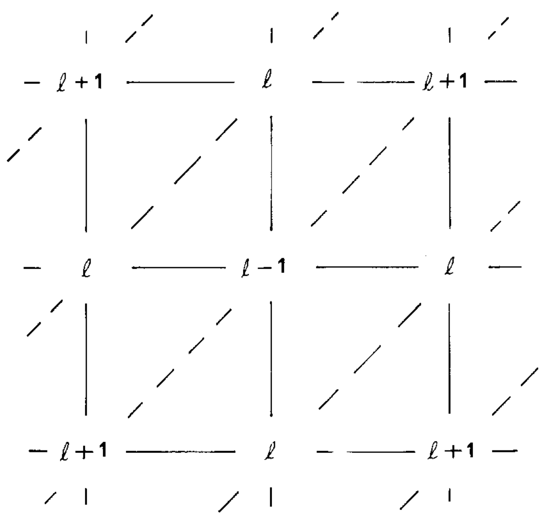
From Fig. 3 and (1.4.2) we have for the Boltzmann weight per pair of sites

$$\begin{aligned} N_l &= \left| E^2(w) \frac{E(x^l w)\, E(x^{l+1}/w)}{E(x^l)\, E(x^{l+1})} \right| \\ Q_l &= \left| x^{1/2} E(w)\, E(x/w) \frac{[E(x^{l-1})\, E(x^{l+1})]^{1/2}}{E(x^l)} \right| \\ R &= \left| wE(x/w) \left[\frac{E(x)\, E(x^2/w)}{E(x/w)\, E(x^2)\, w^{\mu/2}} \right]^{2/(r-2)} \right| \end{aligned} \tag{1.5.4}$$

where we have set the constant v in (1.4.2) equal to 1. Note N_l, Q_l, and R are all unchanged by replacing x^l by $x^l y$. It is therefore convenient to regard

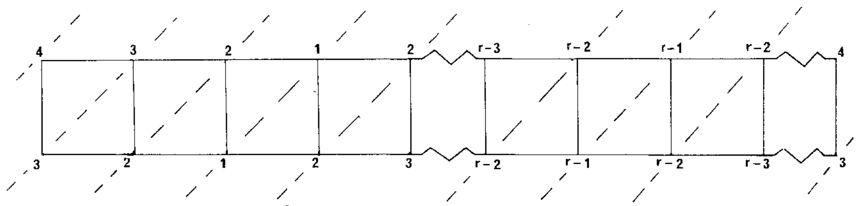


(a) $N_\ell = |\gamma_\ell \;\; \delta_{\ell+1}|$



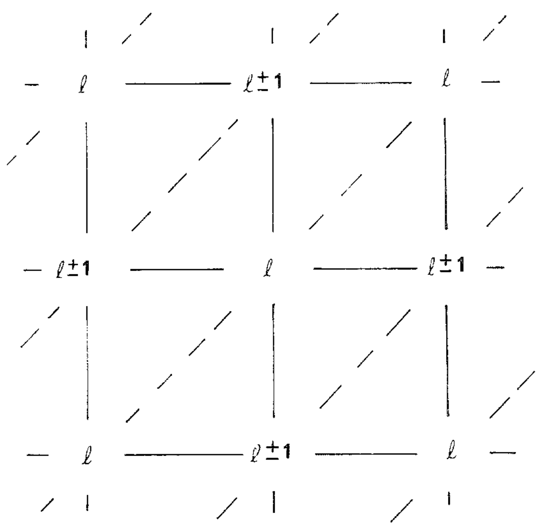
(b) $Q_\ell = |\alpha_\ell \;\; \beta_\ell|$

Fig. 3. The ground states and their associated Boltzmann weights per pair of sites. Dotted lines indicate the heights all have the same values along that diagonal, and in (c) jagged lines indicate the heights successively take every integer value between the heights on each end of the jagged lines.



$R = |(a_2^2 a_3^2 \ldots a_{r-2}^2 \delta_2 \gamma_{r-2})^{1/(r-2)}|$

(c)



$S_\ell = 2N_\ell$

(d) provided $N_\ell = N_{\ell-1} = Q_\ell$

Fig. 3 (*continued*)

x^l as an independent variable (denoted by z, say) so we can restrict our attention to the interval

$$|y| < |z| \leqslant 1 \tag{1.5.5}$$

Note in general, for given values of y and x, z is a discrete variable which can assume the values

$$x^l y^n, \qquad l = 1, 2, \ldots, r-2 \tag{1.5.6}$$

where n is any integer.

From the product definition of the E function we easily calculate the values of N_l, Q_l, and R (for the latter two we have to consider separately the cases $y < x^2$ and $y > x^2$). For the disordered ground state S_l to be a candidate, equality between N_l, N_{l-1}, and Q_l is required (recall Fig. 3). From the calculations of N_l and Q_l we see this is only possible in the portion of Regime III_G corresponding to Regime I and IV when $l = r/2$ (r even).

Comparison of the values of N_l, Q_l, R, and S_l thus obtained shows N_l is a ground state, provided $y/x < |z| < 1$, which from (1.5.6) is equivalent to requiring for some $l = 1, 2, \ldots, r-2$

$$0 < n + l\mu/r < 1 - \mu/r \tag{1.5.7}$$

where for each regime μ is defined by (1.4.11), and n is any integer. This equation has each $l = 1, \ldots, (r-2)$ as solutions, excluding those l in the intervals

$$\left[\frac{r}{\mu} - 1, \frac{r}{\mu}\right], \left[\frac{2r}{\mu} - 1, \frac{2r}{\mu}\right], \ldots, \left[\frac{(\mu-1)r}{\mu} - 1, \frac{(\mu-1)r}{\mu}\right] \tag{1.5.8}$$

where [ ] denotes here (and will do so throughout) the integer part function. In Regimes I and IV, when r is even, the configuration $S_{r/2}$ is also a ground state.

In Regime II with $s = 1$, Regime III with $s = r - 1$, Regime IV with $s = (r-1)/2$ and in Regimes VIII and X the inequality (1.5.7) is not satisfiable (all these cases occur when $\mu = r - 1$). We find the ground states are Q_l ($l = 2, \ldots, r-2$) for Regimes VIII and X, and R in the remaining regimes.

The ground states are summarized in Table I.

1.6. P_a for the Large but Finite Lattice

We want to evaluate the P_a, that is the probability the center height l_1 has height a. The lattice $\mathscr{L}$ is taken to be planar and finite. Suppose for definiteness the center site lies on the X sublattice (recall Fig. 2). As noted in A, the P_a will depend on the particular ground state, so that the number of distinctive ground states will equal the number of distinctive P_a is a given regime. The ground states can be naturally divided into two classes according to whether the even or odd heights lie on the X sublattice (which we call even and odd ground states, respectively).If the ground state is even (odd) we clearly have

$$P_a = 0, \qquad \text{unless } a \text{ is even (odd)} \tag{1.6.1}$$

In calculating the P_a, the ground state is determined by fixing the values of the heights on the outer two sites of each end of each row and

Table I

Regime	N_l[a]	Q_l	R	S_l	Total ground states[b]
I, r even	$2s-2$	0	0	1	$4s-2$
I, r odd	$2s-1$	0	0	0	$4s-2$
II, $s \neq 1$	$s-1$	0	0	0	$2s-2$
II, $s=1$	0	0	1	0	$2r-4$
III, $s \neq (r-1)$	$r-s-1$	0	0	0	$2(r-s-1)$
III, $s=r-1$	0	0	1	0	$2r-4$
IV, r even	$r-2s-2$	0	0	1	$2(r-2s-1)$
IV, r odd, $s \neq (r-1)/2$	$r-2s-1$	0	0	0	$2(r-2s-1)$
IV, $s=(r-1)/2$	0	0	1	0	$2r-4$
VIII	0	$r-3$	0	0	$4r-12$
X	0	$r-3$	0	0	$4r-12$

[a] The entry under the headings N_l, Q_l, and S_l gives the total number of distinct ground states of that type, excluding those obtained from translations.

[b] The "total ground states" column gives the total number of ground states for the regime including those obtained from translations.

column. If the boundary heights correspond exactly to those of a particular ground state, then the P_a corresponds to that ground state. If the boundary heights do not correspond to a particular ground state, then the P_a will correspond to the ground state which has the greatest number of sites with heights at the ground state value. For example, in Regime IV with $r=5$, we see from Section 1.5 that the ground states are L_1 and L_3. If we choose as the boundary heights 2 and 3, then the P_a would correspond to L_1 (see Fig. 4).

As in A, we calculate the P_a using the corner transfer matrix technique. This is done in the Appendix, but the method fails for the unphysical Regimes VIII and X, so we shall from now on restrict attention to Regimes I to IV. It suffices to calculate the P_a in Regime III_G, with x given by (1.4.4), y given by (1.4.7), and μ restricted to the values (1.4.11).

Define the function $c(k, l, m)$ by

$$\begin{gathered} c(l, l-1, l) = [l\mu/r] \\ c(l, l+1, l) = -[l\mu/r] \\ c(l-1, l, l+1) = c(l+1, l, l-1) = 1/2 \end{gathered} \tag{1.6.2}$$

We then find in all cases except $\mu = r-1$

$$P_a = S^{-1} E(x^a, y)\, D_m(a, b, c'; x^2) \tag{1.6.3}$$

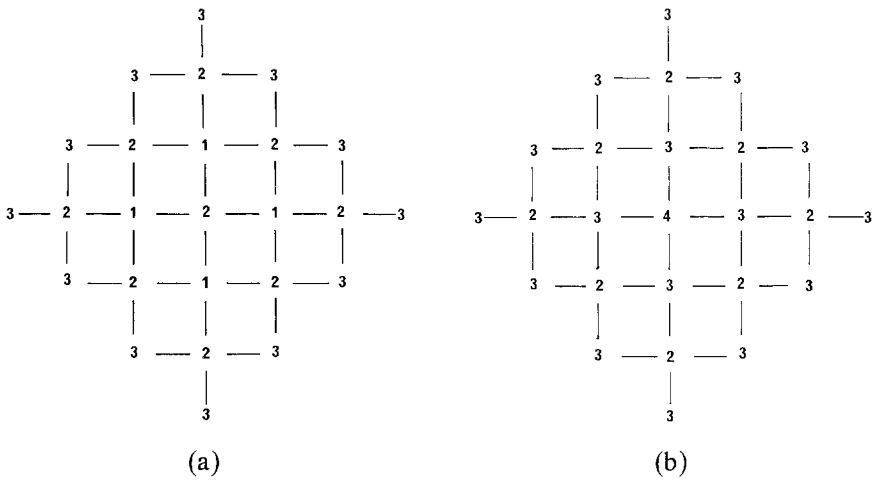


Fig. 4. With the boundary heights fixed at 2 and 3, the remaining heights have been fixed at an L_1 ground state in (a) and an L_3 ground state in (b). More sites of that lattice have heights corresponding to the L_1 ground state.

for $a = 1, 2,..., r-1$ where

$$D_m(a, b, c'; q) = \sum_{l_2,...,l_m} q^{\phi(\mathbf{l})} \tag{1.6.4}$$

$$\phi(\mathbf{l}) = c(l_1, l_2, l_3) + 2c(l_2, l_3, l_4) + \cdots + mc(l_m, l_{m+1}, l_{m+2}) \tag{1.6.5}$$

$$S = \sum_{a=1}^{r-1} {}^*E(x^a, y)\, D_m(a, b, c'; x^2) \tag{1.6.6}$$

Here $\mathbf{l} = \{l_1, l_2,..., l_{m+2})$ is a set of integer heights satisfying the restrictions

$$1 \leqslant l_j \leqslant r-1, \qquad |l_{j+1} - l_j| = 1 \tag{1.6.7}$$

for $j \geqslant 1$. The summation in (1.6.4) is over all allowed values of $l_2,..., l_m$, l_1, l_{m+1}, l_{m+2} being fixed at the values

$$l_1 = a, \qquad l_{m+1} = b, \qquad l_{m+2} = c' \tag{1.6.8}$$

Thus a, b, c' must lie in the range $1, 2,..., r-1$ and must satisfy

$$|b - c'| = 1, \qquad m + a - b = \text{even integer} \tag{1.6.9}$$

The heights $l_1,..., l_{m+2}$ correspond to the heights of sites on the center row of $\mathscr{L}$, starting at the center site 1 and moving right to the boundary. Thus l_{m+1}, l_{m+2} i.e., b, c' are the fixed boundary heights. In (1.6.6) the asterisk indicates the sum is restricted to either all even or all odd integers from 1 to $r-1$ (corresponding to the even and odd ground states, respectively).

The case $\mu = r-1$ requires separate treatment. Instead of considering Regime III_G we consider the region

$$1 < |w| < |y/x^2|^{1/2}, \qquad y = x^r \tag{1.6.10}$$

which when $x, y > 0$ is Regime II with $s = 1$. Let us denote this region by Regime II_G. The transformation (1.4.9) maps the weights (1.4.2) defined in Regime II_G into Regime III_G with $\mu = r-1$. Thus, in an obvious notation

$$P_a^{\mathrm{III}_G}(x, y, r-1) = P_a^{\mathrm{II}_G}(y/x, y, 1), \qquad y^{r-1} = x^r \tag{1.6.11}$$

We find in the Appendix

$$P_a^{\mathrm{II}_G}(x, y, 1) = T^{-1} u_a D_m(a, b, c'; x^{2-r}) \tag{1.6.12}$$

where

$$u_a = x^{(a^2 - ra)/4} E(x^a, y) \tag{1.6.13}$$

$$T = \sum_{a=1}^{r-1}{}^{*} u_a D_m(a, b, c'; x^{2-r}) \tag{1.6.14}$$

The function D_m is defined by (1.6.4) with $\mu = 1$, and the asterisk in (1.6.14) has the same meaning as in (1.6.6).

2. EVALUATION OF THE PROBABILITIES P_a

2.1. Comments on the Method

The expression (1.6.3) for the P_a in terms of the m-fold sum D_m is a generalization of the corresponding sums in A (and of course includes those sums). In A a generalization of I. Schur's proof of the Rogers–Ramanujan identities[4] was used to transform the P_a in terms of the Gaussian polynomials. This form was then found suitable for taking the limit $m \to \infty$. We again find a similar approach suffices.

In Regime II of A (Regime II of this paper with $s = 1$) this limit was not easy to obtain: it was necessary to use the theory of well-poised hypergeometric series. This will also be the case here in Regime III when $s = r-1$, and in Regime IV when $s = (r-1)/2$. However from (1.4.11) these regimes are simply related to Regime II when $s = 1$, so we use the results obtained in A for this region.

2.2. Gaussian Polynomials

The Gaussian polynomials are defined as

$$\begin{bmatrix} N \\ M \end{bmatrix} = \prod_{j=1}^{M} \frac{1-q^{N-M+j}}{1-q^j}, \qquad 0 \leqslant M \leqslant N$$
$$= 0 \qquad\qquad \text{otherwise} \tag{2.2.1}$$

We will require the two recurrences (p. 35 of Ref. 7)

$$\begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} N-1 \\ M \end{bmatrix} = q^{N-M} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} \tag{2.2.2}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} = q^{M} \begin{bmatrix} N-1 \\ M \end{bmatrix} \tag{2.2.3}$$

the symmetry property

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N \\ N-M \end{bmatrix} \tag{2.2.4}$$

and the limiting behavior (for $|q| < 1$)

$$\lim_{N,M\to\infty} \begin{bmatrix} N+M \\ M \end{bmatrix} = (Q(q))^{-1} \tag{2.2.5}$$

where

$$Q(q) = \prod_{n=1}^{\infty} (1-q^n) \tag{2.2.6}$$

2.3. The P_a in Terms of the Gaussian Polynomials

Let us write the function $D_m(a, b, c; q)$ simply as $D_m(a, b, c)$. Here we express the functions $D_m(a, b, b+1)$ and $D_m(a, b, b-1)$ in terms of the Gaussian polynomials. From (1.6.2)–(1.6.5) the function D_m can be totally defined by (a) the recurrences,

$$D_m(a, b, b+1) = q^{m[(b+1)\mu/r]} D_{m-1}(a, b+1, b) + q^{m/2} D_{m-1}(a, b-1, b) \tag{2.3.1}$$

$$D_m(a, b, b-1) = q^{m/2} D_{m-1}(a, b+1, b) + q^{-m[(b-1)\mu/r]} D_{m-1}(a, b-1, b) \tag{2.3.2}$$

(b) the boundary conditions,

$$D_m(a, b-1, b) = 0 \qquad \text{when } b = 1 \tag{2.3.3}$$

$$D_m(a, b+1, b) = 0 \qquad \text{when } b = r-1 \tag{2.3.4}$$

(c) the initial conditions

$$D_0(a, b, b+1) = D_0(a, b, b-1) = \delta_{a,b} \tag{2.3.5}$$

To investigate the recurrences (2.3.1) and (2.3.2) we must consider values of the function $[b\mu/r]$, $1 \leqslant b \leqslant r-1$. Since $1 \leqslant \mu \leqslant r-1$ this function is zero when $b = 1$ at least. The function then increases in integer steps with the largest assumed value being $\mu - 1$. Divide the b values into sets I_k such that

$$I_k = \{b : [b\mu/r] = k,\ 1 \leqslant b \leqslant r-1\}, \qquad k = 0, 1, \ldots \mu - 1 \tag{2.3.6}$$

Note since $\mu < r$, each I_k is nonempty. Denote the smallest element of the set I_k by e_k and the largest element by e'_k. We can then classify the values of the functions $[(b+1)\mu/r]$ and $[(b-1)\mu/r]$. We have

$$L_k \equiv \{b : [(b+1)\mu/r] = k,\ 1 \leqslant b \leqslant r-2\} = (I_k U\{e'_{k-1}\}) - \{e'_k\} \tag{2.3.7}$$

$$M_k \equiv \{b : [(b-1)\mu/r] = k,\ 2 \leqslant b \leqslant r-1\} = (I_k U\{e_{k+1}\}) - \{e_k\} \tag{2.3.8}$$

Our strategy is to consider the function D_m as lying in different domains, according to the value of $[(b+1)\mu/r]$ and $[(b-1)\mu/r]$. We say the function $D_m(a, b, b+1)$ lies in domain k if $b \in L_k$, while we say $D_m(a, b, b-1)$ lies in domain k if $b \in M_k$. In both cases we indicate the domain by writing $D_m = D_m^{(k)}$.

Then we can write the recurrences (2.3.1) and (2.3.2) as the 2μ recurrences

$$D_m^{(k)}(a, b, b+1) = q^{km} D_{m-1}^{(k)}(a, b+1, b) + q^{m/2} D_{m-1}^{(k)}(a, b-1, b)$$
$$b \in L_k - \{e'_{k-1}\}, \qquad k = 0, 1, \ldots, \mu - 1 \tag{2.3.9}$$

$$D_m^{(k)}(a, b, b-1) = q^{m/2} D_{m-1}^{(k)}(a, b+1, b) + q^{-km} D_{m-1}^{(k)}(a, b-1, b)$$
$$b \in M_k - \{e_{k+1}\}, \qquad k = 0, 1, \ldots, \mu - 1 \tag{2.3.10}$$

$$D_m^{(k+1)}(a, e'_k, e'_k + 1) = q^{(k+1)m} D_{m-1}^{(k)}(a, e'_k + 1, e'_k) + q^{m/2} D_{m-1}^{(k)}(a, e'_k - 1, e'_k) \tag{2.3.11}$$

$$D_m^{(k)}(a, e_{k+1}, e_{k+1} - 1) = q^{m/2} D_{m-1}^{(k+1)}(a, e_{k+1} + 1, e_{k+1}) + q^{-km} D_{m-1}^{(k+1)}(a, e_{k+1} - 1, e_{k+1}) \tag{2.3.12}$$

where in the last two recurrences $k=0, 1,..., \mu-2$ (thus if $\mu=1$ these recurrences are not present) and we take $\{e'_{k-1}\}$ and $\{e_{k+1}\}$ to be null if $k=0$, $\mu-1$, respectively. The recurrence (2.3.11) can be simplified if we substitute $b=e'_k$ in (2.3.10), multiply both sides of the equation by $q^{km+m/2}$, and then substitute in (2.3.11). This gives in place of (2.3.11)

$$D_m^{(k)}(a, e'_k, e'_k-1)=q^{-km-m/2}D_m^{(k+1)}(a, e'_k, e'_k+1) \qquad (2.3.13)$$

Similarly, replace k by $k+1$ in recurrence (2.3.9), substitute $b=e_{k+1}$, multiply both sides of the equation by $q^{-km-m/2}$, and then substitute in (2.3.12). This gives in place of (2.3.12)

$$D_m^{(k)}(a, e_{k+1}, e_{k+1}-1)=q^{-km-m/2}D_m^{(k+1)}(a, e_{k+1}, e_{k+1}+1) \qquad (2.3.14)$$

The equations (2.3.13) and (2.3.14) are regarded as boundary conditions between the domains. Note that a sufficient condition for both these equations to hold is

$$D_m^{(k)}(a, b, b-1)=q^{-km-m/2}D_m^{(k+1)}(a, b, b+1) \qquad (2.3.15)$$

for all integers b. We will in fact establish this more general property. Hence the $D_m^{(k)}$ (and thus the D_m) are totally defined by the recurrences (2.3.9) and (2.3.10), the boundary conditions (2.3.3), (2.3.4), and (2.3.15), and the initial condition (2.3.5).

By analogy with the form of the solutions obtained in A we sought solutions of the form

$$\sum_{\lambda} q^{\gamma\lambda^2+\alpha(a,b,b\pm 1)\lambda+\beta(a,b,b\pm 1)+m\tau(a,b,b\pm 1)} \begin{bmatrix} m \\ \frac{m+a-b}{2}-r\lambda \end{bmatrix} \qquad (2.3.16)$$

where the γ, α, β, and τ were to be determined. We thus derived the following result.

Theorem 2.3.1. For $m \geqslant 0$, $1 \leqslant a, b < r$, $m+a-b$ an even integer

$$D_m^{(k)}(a, b, b\pm 1)=q^{a(a-1)/4+m\tau(a,b,b\pm 1)}(g^{(k)}(a, b, b\pm 1)-g^{(k)}(-a, b, b\pm 1)) \qquad (2.3.17a)$$

where

$$\tau(a, b, b\pm 1)=\pm k/2 \qquad (2.3.17b)$$

$$g^{(k)}(a, b, b\pm 1)=\sum_{\lambda=-\infty}^{\infty} q^{r(r-\mu)\lambda^2+\alpha(a,b,b\pm 1)\lambda+\beta(a,b,b\pm 1)} \begin{bmatrix} m \\ \frac{m+a-b}{2}-r\lambda \end{bmatrix} \qquad (2.3.17c)$$

$$\alpha(a, b, b \pm 1) = r((b + b \pm 1 - 1)/2 - k) - a(r - \mu) \tag{2.3.17d}$$

$$\beta(a, b, b + 1) = b(b - 1)/4 - (a + k - 1)b/2 + ak/2 \tag{2.3.17e}$$

$$\beta(a, b, b - 1) = b(b - 1)/4 - (a + k)b/2 + a(k + 1)/2 \tag{2.3.17f}$$

Proof. From the discussion above if suffices to verify (2.3.9), (2.3.10), (2.3.15), (2.3.3), (2.3.4), and (2.3.5).

To verify (2.3.9) substitute for $D_m^{(k)}$ (2.3.17) and then use (2.2.3) in the form

$$\begin{bmatrix} m-1 \\ \frac{m+a-b}{2} - r\lambda - 1 \end{bmatrix} = \begin{bmatrix} m \\ \frac{m+a-b}{2} - r\lambda \end{bmatrix} - q^{(1/2)(m+a-b)-r\lambda} \begin{bmatrix} m-1 \\ \frac{m+a-b}{2} - r\lambda \end{bmatrix} \tag{2.3.18}$$

to replace the Gaussian polynomial in the function $D_{m-1}^{(k)}(a, b + 1, b)$. This shows (2.3.9) is true for all b.

Similarly (2.3.10) follows immediately, if we use (2.2.2) in the form

$$\begin{bmatrix} m-1 \\ \frac{m+a-b}{2} - r\lambda - 1 \end{bmatrix} = q^{(a-b-m)/2-r\lambda} \left(\begin{bmatrix} m \\ \frac{m+a-b}{2} - r\lambda \end{bmatrix} - \begin{bmatrix} m-1 \\ \frac{m+a-b}{2} - r\lambda \end{bmatrix} \right) \tag{2.3.19}$$

to replace the Gaussian polynomial in the function $D_m^{(k)}(a, b + 1, b)$.

To verify (2.3.15) we merely substitute (2.3.18). The result follows without any manipulation.

To check (2.3.3) consider $F^{(k)}(a, 0, 1)$ as defined by (2.3.19). Replacing λ by $-\lambda$, then using the property of the Gaussian polynomial (2.2.4) shows

$$g^{(k)}(a, 0, 1) = g^{(k)}(-a, 0, 1) \tag{2.3.20}$$

Hence from (2.3.17), (2.3.3) is satisfied. For the boundary condition (2.3.4) consider $F^{(k)}(a, r, r - 1)$ as defined by (2.3.17). Replacing λ by $-(\lambda + 1)$, then using (2.2.4) shows

$$g^{(k)}(a, r, r - 1) = g^{(k)}(-a, r, r - 1) \tag{2.3.21}$$

which from (2.3.17) establishes (2.3.14).

It remains to verify (2.3.5). When $m=0$ the only nonzero term in (2.3.17) is the $\lambda=0$ term, since from (2.2.1)

$$\begin{bmatrix} 0 \\ N \end{bmatrix} = \delta_{0,N} \tag{2.3.22}$$

Hence, since $1 \leqslant a,\ b \leqslant r-1$ we have from (2.3.17) when $m=0$

$$g^{(k)}(-a, b, b \pm 1) = 0 \tag{2.3.23}$$

$$g^{(k)}(a, b, b \pm 1) = q^{-a(a-1)/4}\,\delta_{a,b} \tag{2.3.24}$$

Substituting (2.3.23) and (2.3.24) in (2.3.17) we establish (2.3.5) and thus the theorem. ∎

2.4. The Large *m* Limit

We require the $m \to \infty$ limit of the P_a (1.6.3). This is equivalent to taking the $m \to \infty$ limit of $q^{\mp km/2} D_m^{(k)}(a, b, b \pm 1)$ provided this limit is nonzero. Let us consider this limit first. Define the function

$$\begin{aligned} F(a, b-k; q^{1/2}) = q^{a(a-1)/4}\,[&q^{-a(b-k)/2} E(-q^{r(b-k)+(r-a)(e-\mu)}, q^{2r(r-\mu)}) \\ &- q^{a(b-k)/2} E(-q^{r(b-k)+(r+a)(r-\mu)}, q^{2r(r-\mu)})] \end{aligned} \tag{2.4.1}$$

and denote by "$\lim_{m\to\infty}$" the limit $m \to \infty$ restricted to those values of m in $D_m(a, b, b \pm 1)$ with the same parity as $a-b$. Then we have the following result.

Theorem 2.4.1. Let $1 \leqslant a \leqslant r-1$, $1 \leqslant b \leqslant r-2$. Then

$$\begin{aligned} &\lim_{m\to\infty} q^{-km/2} D_m^{(k)}(a, b, b+1) \\ &\qquad = (Q(q))^{-1} q^{b(b-1)/4-(k-1)b/2} F(a, b-k; q^{1/2}) \end{aligned} \tag{2.4.2}$$

where

$$k = [\mu(b+1)/r] \tag{2.4.3}$$

and

$$\begin{aligned} &\lim_{m\to\infty} q^{km/2} D_m^{(k)}(a, b+1, b) \\ &\qquad = [Q(q)]^{-1} q^{b(b+1)/4-k(b+1)/2} F(a, b-k; q^{1/2}) \end{aligned} \tag{2.4.4}$$

where

$$k = [\mu b/r] \tag{2.4.5}$$

The function $Q(q)$ is defined by (2.2.6).

Proof. Firstly, note the values of k follows from the definition of the $D_m^{(k)}$ given in Section 2.3. The theorem follows immediately after taking the limit inside the summation (2.3.19), then using the formula for the limiting form of the Gaussian polynomials (2.2.5). The resulting expression is identified with (2.4.1) using the series expansion of the E function (1.4.1). ∎

Next we consider the conditions under which the function F vanishes. This occurs when

$$(b-k)=l(r-\mu)/\mu \tag{2.4.6}$$

for some integer l. To see this substitute (2.4.6) for $(b-k)$ in the second term of (2.4.1). Then in the series definition of the E function (1.4.1) replace λ by $\lambda+m$. The second term is then seen to be equal to the first in (2.4.1) and thus F vanishes.

Note from (2.4.3) the quantity b–k in $D_m^{(k)}(a,b,b+1)$ is monotone, taking the value 0 to $r-\mu-1$ inclusive. Thus is this case the condition (2.4.6) is only satisfied when

$$b-k=0 \tag{2.4.7}$$

From (2.4.5), $b-k$ in $D_m^{(k)}(a,b+1,b)$ is again monotone, taking the values 0 to $r-\mu$ inclusive. Hence is this case the condition (2.4.6) is satisfied when

$$b-k=0 \qquad \text{or} \qquad b-k=r-\mu \tag{2.4.8}$$

To take the $m\to\infty$ limit of the P_a under the conditions (2.4.7) and (2.4.8) we expand $q^{\mp km/2}D_m^{(k)}(a,b,b\pm 1)$ in powers of $q^{-m/2}$. This is done via the special case of Lemma 2.4.8 of A [choose $B=(a-b)/2-r\lambda$, $b=b-rl$]

$$\lim_{m\to\infty} q^{-m/2}\left\{\begin{bmatrix} m \\ \frac{m+a-b}{2}-r\lambda\end{bmatrix}-\begin{bmatrix} m \\ \frac{m+a+b}{2}-r(\lambda+l)\end{bmatrix}\right\}$$
$$=q^{-b/2}(1-q^{(b-rl)})(q^{-a/2+r(\lambda+l)}-q^{a/2+1-r\lambda})/(1-q)\,Q(q) \tag{2.4.9}$$

where $l=0$ or 1 according to $(b-k)$ equaling 0 or $r-\mu$, respectively. Using (2.4.9) in (2.3.17) we readily derive the following result.

Theorem 2.4.2. If $b-k=0$,

$$\lim_{m\to\infty} q^{-m/2}(q^{-km/2}D_m^{(k)}(a,b,b+1))$$
$$=(q^{b(b-1)/4-kb/2}(1-q^b)/(1-q)\,Q(q))\,F(a,1;q^{1/2}) \tag{2.4.10}$$

and

$$\lim_{m\to\infty} q^{-m/2}(q^{km/2}D_m^{(k)}(a,b+1,b))$$
$$=(q^{b(b-1)/4-k(b+1)/2-1/2}(1-q^{b+1})/(1-q)\,Q(q))\,F(a,1;q^{1/2}) \tag{2.4.11}$$

If $b - k = r - \mu$,

$$\lim_{m\to\infty} q^{-m/2}(q^{km/2} D_m^{(k)}(a, b+1, b))$$

$$= (q^{b(b-1)/4 - k(b+1)/2 - 1/2}(1 - q^{b+1-r})/(1-q)\, Q(q))\, F(a, r-\mu-1; q^{1/2}) \tag{2.4.12}$$

Theorems 2.4.1 and 2.4.2 provide the limiting value of the P_a for all allowed values of μ as given by (1.4.11), except $\mu = (r-1)$, in which case $F(a, 0; q) = F(a, 1; q) = 0$. From (1.4.11) the case $\mu = r - 1$ corresponds to Regime II when $s = 1$, Regime III when $s = r - 1$ and Regime IV when $s = (r-1)/2$. As noted in Section 2.1 we can use the results obtained in A for Regime II when $s = 1$ to write down the P_a in these cases.

Thus substituting Theorems 2.4.1 and 2.4.2 in (1.6.3), provided $\mu \neq (r-1)$ we have computed P_a in the $m \to \infty$ limit.

Theorem 2.4.3. Consider the portion of Regime III_G corresponding to Regimes I, II, III, and IV, excluding the case $\mu = r - 1$. Suppose we impose the boundary condition of heights $b, b+1$ $(1 \leqslant b \leqslant r-2)$ on the second end and end sites, respectively, of each row. Define k by (2.4.3). Then if $b - k \neq 0$

$$P_a = E(x^a, y)\, F(a, b-k; x) \Big/ \sum_{a=1}^{r-1}{}^{*}\, E(x^a, y)\, F(a, b-k; x) \tag{2.4.13}$$

while if $b - k = 0$

$$P_a = E(x^a, y)\, F(a, 1; x) \Big/ \sum_{a=1}^{r-1}{}^{*}\, E(x^a, y)\, F(a, 1; x) \tag{2.4.14}$$

Suppose instead we impose the boundary condition of heights $b+1, b$ $(1 \leqslant b \leqslant r-2)$ on the second end and end sites, respectively, of each row, and now define k by (2.4.5). Then if $b - k \neq 0$, $r - \mu$ the P_a is given by (2.4.13) and if $b - k = 0$ by (2.4.14). If $b - k = r - \mu$ we have

$$P_a = E(x^a, y)\, F(a, r-\mu-1; x) \Big/ \sum_{a=1}^{r-1}{}^{*}\, E(x^a, y)\, F(a, r-\mu-1; x) \tag{2.4.15}$$

It remains to consider the case $\mu = r - 1$, which from (1.6.11) and (1.6.12) is equivalent to considering the limiting behavior of the sum D_m with $\mu = 1$ and $|q| > 1$. This limit has been calculated in A (Theorem 2.6.8). Hence substituting the results of A in (1.6.12) we have evaluated the P_a in Regime II_G, which from (1.6.11) gives the P_a in Regime III_G with $\mu = r - 1$. We will present the results for Regime II_G.

Theorem 2.4.4. Define the function

$$\Phi_a(z)=\frac{E(q^a,q^r)\,E(-z,q)(Q(q^r))^3}{E(-z,q^r)\,E(-q^az,q^r)(Q(q))^3} \tag{2.4.16}$$

This is an analytic function in the domain $0<|z|<\infty$ and so has the Laurent expansion

$$\Phi_a(z)=\sum_{k=-\infty}^{\infty}\eta_{a,k}z^k \tag{2.4.17}$$

Define $\hat{\eta}_{a,k}$ by

$$\eta_{a,k}=q^{[rk(k+1)/2-ak]/(r-2)}\hat{\eta}_{a,k} \tag{2.4.18}$$

Consider Regime II_G given by (1.6.10). Then in terms of the $\hat{\eta}_{a,k}$ with $q=x^{r-2}$

$$P_a=E(x^a,y)\hat{\eta}_{a,(a+j)/2}\Big/\sum_{a=1}^{r-1}{}^{*}\,E(x^a,y)\,\hat{\eta}_{a,(a+j)/2} \tag{2.4.19}$$

If the boundary condition of the second end and end site of each row is heights b and $b+1$, respectively, then the parameter j in (2.4.19) has the values

$$(m-b)\bmod(2r-4) \tag{2.4.20}$$

while if they are $b+1,b$ then j has the values

$$-(m+b+1)\bmod(2r-4) \tag{2.4.21}$$

(thus for the limit $m\to\infty$ to be well defined we must, in the limiting procedure, hold either (2.4.20) or (2.4.21) fixed). Furthermore we require j to have the same parity as a.

2.5. Number of Ground States from the P_a

We noted in Section 1.6 that the number of distinctive ground states will equal the number of distinctive P_a in the given regime. Let us now check that assertion. First suppose the center height a is even, so we are considering even ground states. Consider the case $\mu\neq(r-1)$. From Theorem 2.4.3 the number of distinct P_a for boundary heights $b+1,b$ is equal to the number of distinct values of $b-k$, minus 1, where k is given by (2.4.3). For boundary heights $b,b+1$ the number of distinct P_a is equal to the number of distinct values of $b-k$, minus 2, where k is given by (2.4.5). In the former case there are $r-\mu$ distinct values, while in the latter there are $r-\mu+1$. Thus in both cases the number of even ground states is $r-\mu-1$. Similarly there are $r-\mu-1$ odd ground states, giving a total number of $2(r-\mu-1)$ distinctive ground states. By writing down the corresponding value of μ in terms of s from (1.4.11) for Regimes I–IV, $\mu \neq (r-1)$ we see this is indeed the number of ground states given by Table I.

When $\mu = r-1$, from Theorem 2.4.4 we see the number of even ground states is given by the number of distinct values of the parameter j, which is $r-2$. Similarly there are $r-2$ odd ground states, giving the total number of distinctive ground states as $2r-4$. This is consistent with the number of ground states given in Table I for Regime II with $s=1$, Regime III with $s=r-1$ and Regime IV with $s=(r-1)/2$ all of which correspond to Regime III_G with $\mu = r-1$.

2.6. Normalization Constant

We have now calculated the large-m limits of the D_m in (1.6.3). We still need to evaluate the normalizations S and T as defined by (1.6.6) and (1.6.14). In A we showed that S could be expressed as a single product of two E functions, and T as a Q function. This is again the case here. Indeed the summation formulas proved in A (Theorems 3.2.1 and 3.2.3) contain the required identities.

In Theorem 3.2.1 of A choose $m=2\mu$, $z=x^{k-b}$ and $\varepsilon=-1$. We then have the identity

$$
\begin{aligned}
S(x,y) \equiv & \sum_{-r<a<r}^{*} x^{a(a-1)/2-(b-k)a} E(x^a, y) \\
& \times E(-x^{2(r-\mu)(r-a)+2r(b-k)}, x^{4r(r-\mu)}) \\
& = 1/2(E(-x^{k-b}, x)\, E(x^{(b-k)}, y/x) \pm E(x^{b-k}, x)\, E(-x^{-(b-k)}, y/x)) \\
& = x^{-(b-k)(b-k+1)/2} E(-x, x^4)\, E(x^{b-k}, y/x)
\end{aligned} \tag{2.6.1}
$$

where to obtain the last line from the second last we have used the simple identities

$$E(x^u, x) = 0 \tag{2.6.2a}$$

$$E(-x^{-u}, x) = 2x^{u(u+1)/2} E(-x, x^4) \tag{2.6.2b}$$

valid for all integers u. The identity (2.6.1) is true for all integers b and k and all real numbers y, x such that

$$y^{\mu} = x^{r}, \qquad 1 \leqslant \mu \leqslant r-1 \tag{2.6.3}$$

Noting (2.6.4) is consistent with (1.4.7), and by grouping together the a and $-a$ terms in the sum $S(x, y)$ using the identity

$$E(x^{-a}, y) = -x^{-a} E(x^a, y) \tag{2.6.4}$$

we see $S(x, y)$ is the required normalization constant in Theorem 2.4.3. We have thus calculated the normalization constant in all cases except $\mu = r - 1$.

Recall the case $\mu = r - 1$ corresponds to Regime II with $s = 1$, Regime III with $s = r - 1$ and Regime IV with $s = (r-1)/2$. In Regime II with $s = 1$, from (2.4.19), (1.6.11), and (1.4.12) we require the sum

$$T_1 = \sum_{a=1}^{r-1}{}^{*} E(x^a, x^r)\, \hat{\eta}_{a,(a+j)/2}(x^{r-2}) \tag{2.6.5}$$

In Regime III with $s = r - 1$ we require

$$T_2 = \sum_{a=1}^{r-1}{}^{*} E(x^{a(r-1)}, x^{r/(r-1)})\, \hat{\eta}_{a,(a+j)/2}(x^{(r-2)/(r-1)}) \tag{2.6.6}$$

while in Regime IV with $s = (r-1)/2$ (r odd) we want

$$T_3 = \sum_{a=1}^{r-1}{}^{*} E((-x^{1/(r-1)})^a, -x^{r/(r-1)})\, \hat{\eta}_{a,(a+j)/2}(-x^{(r-2)/(r-1)}) \tag{2.6.7}$$

The values of x in these sums is given by (1.4.4) and (1.4.5) or (1.4.4) and (1.4.6) depending on the regime. Further, we have written $\hat{\eta}_{a,k} = \hat{\eta}_{a,k}(q)$.

In A (Theorem 3.2.2) we proved

$$\sum_{a=1}^{r-1}{}^{*} E(x^a, x^r)\, \hat{\eta}_{a,(a+j)/2}(x^{r-2}) = Q(x^{r-2}) \tag{2.6.8}$$

where the function Q is defined by (2.2.6). This evaluates T_1 immediately, T_2 by replacing x with $x^{1/(r-1)}$ and T_3 by replacing x with $-x^{1/(r-1)}$ (and noting r is odd).

2.7. Final Results

We can now write down the expression for the $P_a^T(r, s, y)$ ($T = \mathrm{I} - \mathrm{IV}$, where we are using the notation of Section 1.4) using their relationship to the P_a in Regime III_G given by (1.4.11).

In Regimes II and III, where $0 < p < 1$, we can define $\varepsilon = \pi K'/K$, $\varepsilon > 0$, so from (1.2.4), (1.4.4), and (1.4.5)

$$p = e^{-\varepsilon}, \qquad x = e^{-4\pi^2 s/r\varepsilon}, \qquad y = e^{-4\pi^2/\varepsilon} \tag{2.7.1}$$

In Regimes I and IV, where $-1 < p < 0$ (ád thus $K' = L' + iK$) we can take $\varepsilon = \pi L'/K$, $\varepsilon > 0$, so from (1.2.4), (1.4.4), and (1.4.6)

$$p = -e^{-\varepsilon}, \qquad x = e^{-2\pi^2 s/r\varepsilon}, \qquad y = -e^{-\pi^2/\varepsilon} \tag{2.7.2}$$

With the notation introduced in (2.7.1) and (2.7.2) we write the arguments of the $P_a^{\mathrm{III}G}$ in (1.4.11) in terms of the variable y and the parameters r and s. Thus

$$\begin{aligned} P_a^{\mathrm{I}}(r, s, y) &= P_a^{\mathrm{III}G}(-|y|^{1-(2s/r)}, y, r-2s) \\ P_a^{\mathrm{II}}(r, s, y) &= P_a^{\mathrm{III}G}(y^{1-(s/r)}, y, r-s) \\ P_a^{\mathrm{III}}(r, s, y) &= P_a^{\mathrm{III}G}(y^{s/r}, y, s) \\ P_a^{\mathrm{IV}}(r, s, y) &= P_a^{\mathrm{III}G}(|y|^{2s/r}, y, 2s) \end{aligned} \tag{2.7.3}$$

In all cases except $\mu = r - 1$ [i.e., Regime II $s = 1$, Regime III $s = r - 1$ and Regime IV $s = (r-1)/2$] $P_a^{\mathrm{III}G}$ is given by Theorem 2.4.3 with the normalization constant given by (2.6.1). In the case $\mu = r - 1$ we have from (1.6.11), Theorem 2.4.4, (2.6.8), and (2.7.3)

$$\begin{aligned} P_a^{\mathrm{II}}(r, 1, y) &= P_a^{\mathrm{III}}(r, r-1, y) \\ &= E(y^{a/r}, y)\, \hat{\eta}_{a,(a+j)/2}(y^{(r-2)/r})/Q(y^{(r-2)/r}) \end{aligned} \tag{2.7.4}$$

$$\begin{aligned} &P_a^{\mathrm{IV}}(r, (r-1)/2, y) \\ &\quad = E(-(|y|^{1/r})^a, y)\, \hat{\eta}_{a,(a+j)/2}(-|y|^{(r-2)/r})/Q(-|y|^{(r-2)/r}) \end{aligned} \tag{2.7.5}$$

3. CRITICAL BEHAVIOR

3.1. The P_a in Terms of the Original Variables

In Section 1.5 we saw how the different regimes are characterized by different numbers and types of ground states. For given values of v and η in the weights (1.2.2) the regimes depend on the sign of the variable p (1.2.4). When $p = 0$ the model is critical (the actual regimes which are coexisting at this point can be determined from Figure 5). Thus p measures the deviation from criticality so it is desirable to obtain expansions of the P_a in terms of this variable. To do this we require the conjugate modulus form of the expressions (2.7.3), (2.7.4), and (2.7.5) which converts the variable y therein to p [recall we use a conjugate modulus identity in Section 1.4 to write the weights (1.2.2) which were functions of p in terms of y].

We will express our results in terms of the Jacobian θ functions[8] defined in terms of the complex variable u, and parameter $q(|q| < 1)$ by

$$\theta_1(u, q^2) = |q|^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} e^{2iu(n+1/2)} \tag{3.1.1}$$

$$\theta_4(u, q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2ium} \tag{3.1.2}$$

$$\theta_2(u, q^2) = \theta_1(\pi/2 + u, q^2), \qquad \theta_3(u, q^2) = \theta_4(\pi/2 + u, q^2) \tag{3.1.3}$$

As in A we have taken q^2 rather than q as the nome since we encounter θ_1 functions with q^2 negative. Since the definition (3.1.1) is a function of q^2 this is unambiguous. However θ_4 is a function of q rather than q^2. This will not cause any problems as we will only encounter θ_4 functions with q positive.

Also it is convenient to define

$$\tilde{\theta}_i(u, q^2) = \tfrac{1}{2} |q|^{-1/4} \theta_i(u, q^2), \qquad i = 1, 2 \tag{3.1.4}$$

The conjugate modulus identities we require are

$$\theta_1(u, e^{-\varepsilon}) = \rho(u, \varepsilon) E(e^{-4\pi u/\varepsilon}, e^{-4\pi^2/\varepsilon}) \tag{3.1.5a}$$

$$\theta_4(u, e^{-\varepsilon}) = \rho(u, \varepsilon) E(-e^{-4\pi u/\varepsilon}, e^{-4\pi^2/\varepsilon}) \tag{3.1.5b}$$

$$\theta_1(u/2, -e^{-\varepsilon/4}) = 2^{1/2} \rho(u, \varepsilon) E(e^{-4\pi u/\varepsilon}, -e^{-4\pi^2/\varepsilon}) \tag{3.1.5c}$$

$$\theta_2(u/2, -e^{-\varepsilon/4}) = 2^{1/2} \rho(u, \varepsilon) E(-e^{-4\pi u/\varepsilon}, -e^{-4\pi^2/\varepsilon}) \tag{3.1.5d}$$

where

$$\rho(u, \varepsilon) = (2\pi/\varepsilon)^{1/2} \exp[(2\pi u - 2u^2 - \pi^2/2)/\varepsilon] \tag{3.1.5e}$$

(see A for references).

We have already applied the conjugate modulus transformation to the function $\hat{\eta}_{a,j}(\exp[-2\pi^2(r-2)/r\varepsilon])$ which occurs in (2.7.4) [Eqs. (3.37)–(3.3.15) of A]. Define

$$\lambda_{a,j} = \left(\frac{r-2}{r}\right)^{1/2} \exp\left[-\frac{\varepsilon}{8} + \frac{r\varepsilon}{24(r-2)} - \frac{(r-2)\pi^2}{6r\varepsilon} + \frac{(r/2-a)^2\pi^2}{2r^2\varepsilon}\right]$$
$$\times \hat{\eta}_{a,j}(\exp[-2\pi^2(r-2)/r\varepsilon]) \tag{3.1.6}$$

$$t = p^{1/(r-2)} = e^{-\varepsilon/(r-2)} \tag{3.1.7}$$

$$F_a(u) = \frac{Q^3(t)\, \tilde{\theta}_1(\pi a/r, t)\, \theta_4(ru, t^r)}{Q^2(t^r)\, \theta_4(u, t)\, \theta_4(u + \pi a/r, t)} \tag{3.1.8}$$

Let F_a have the Fourier expansion

$$F_a(u) = \sum_{n=-\infty}^{\infty} f_{a,n} e^{2inu} \tag{3.1.9}$$

Then $\lambda_{a,j}$, and thus from (3.1.6) $\hat{\eta}_{a,j}$, is given by

$$\lambda_{a,j} = \frac{4}{r} \sum_{n=0}^{r-3} \exp\left[-2\pi i \left(j + \frac{1}{2} - \frac{a}{r}\right) \Big/ (r-2)\right] t^{-n^2/(2r-4)} f_{a,n} \quad (3.1.10)$$

The same approach suffices to convert the function $\hat{\eta}_{a,j}(-\exp[-\pi^2(r-2)/r\varepsilon])$ occurring in (2.7.5). Define

$$\gamma_{a,j} = \left(\frac{r-2}{r}\right)^{1/2} \exp\left[\frac{\varepsilon r}{24(r-2)} - \frac{\pi^2(r-2)}{24\varepsilon r} + \frac{\pi^2}{2\varepsilon r^2}\left(a - \frac{r}{2}\right)^2\right]$$
$$\times \hat{\eta}_{a,j}\left(-\exp\left[-\frac{\pi^2(r-2)}{r\varepsilon}\right]\right) \quad (3.1.11)$$

$$G_a(u) = \frac{\tilde{\theta}_\tau(\pi a/2r, -t)\, \theta_2(ur, -t^r)\, Q^3(-t)}{\tilde{\theta}_{3-\tau}(u + \pi a/2r, -t)\, \tilde{\theta}_2(u, -t)\, Q^2(-t^r)} \quad (3.1.12)$$

where t is defined by (3.1.7) and $\tau = 1$ if a is even and $\tau = 2$ if a is odd. Let G_a have the Fourier expansion

$$G_a(u) = \sum_{m=-\infty}^{\infty} g_{m,a}\, e^{2iu(m+1/2)} \quad (3.1.13)$$

and denote

$$Y = (t^{-1/8(r-2)}/r) \sum_{m=0}^{r-3} g_{2m,a}(-1)^m t^{-m(2m+1)/(r-2)}$$
$$\times \exp\{-\pi i(4m+1)[(r-1)/2 + j - a/r]/2(r-2)\} \quad (3.1.14)$$

The following the same procedure as used in A to derive (3.1.10) we find

$$\gamma_{a,j} = \begin{cases} (1)^{-j/2} Y, & a \text{ even, } \tfrac{1}{2}(r-1) \text{ even} \\ i(-1)^{j/2} Y, & a \text{ even, } \tfrac{1}{2}(r-1) \text{ odd} \\ i(-1)^{-j/2} Y, & a \text{ odd, } \tfrac{1}{2}(r-1) \text{ even} \\ (-1)^{j/2} Y, & a \text{ odd, } \tfrac{1}{2}(r-1) \text{ odd} \end{cases} \quad (3.1.15)$$

We are now in a position to apply the conjugate modulus transformations (3.1.5) to the P_a. However first note the P_a have been defined in terms of the boundary conditions, which we know determines the underlying ground state. It is more convenient to label the P_a explicitly by the underlying ground state. Consider the P_a in Regime III_G for $\mu \neq r-1$ as given by Theorem 2.4.3. From Section 2.5 the ground state can be labeled by the value of $b-k$ in the function $F(a, b-k; x)$, there being as many distinct values of $b-k$ as ground states. Let us denote $b-k$ by d. Then form the discussion on the ground states contained between equations (1.6.1) and (1.6.2) of the text it follows d labels the allowed ground states in order from lowest to highest (recall from Section 1.5 that for $\mu \neq r-1$ the ground states are of the type N_l or S_l which have a natural ordering according to the value of l).

A reformulation of the results in the case of $\mu = r-1$, when the ground state is of type R (recall Section 1.5) has been given by Huse.[3] We want to write

$$P_a = P_{a,n}^{(J)} \tag{3.1.16}$$

where $P_{a,n}^{(J)}$ is the probability of finding the height $l_i = a$ at a given site i in the interior of the system. To do this consider a unit cell of the ground state R which can be chosen to be the $(2r-4)$ adjacent sites in a row, numbered $n = 0, 1,..., (2r-5)$ with the odd numbered sites on the X sublattice. The $(r-2)$ odd (even) ground states and their corresponding phases may then be numbered by the odd (even) integers $0 \leqslant J \leqslant (2r-5)$, so that in each ground state the height at site J in the unit cell is equal to 1. To rewrite the expressions (2.7.4) and (2.7.5) in terms of this labeling system we merely replace the j therein by $J-n-1$ for odd ground states, and by $J-n$ for even ground states, where in both cases n must have the same parity as a. Denoting the P_a for the odd and even ground states by ${}^{(\mathrm{odd})}P_a$ and ${}^{(\mathrm{ev.})}P_a$, respectively, we clearly have

$${}^{(\mathrm{odd})}P_{a,n}^{(J)} = {}^{(\mathrm{ev.})}P_{a,n-1}^{(J-1)} \tag{3.1.17}$$

Hence is suffices to write down the ${}^{(\mathrm{odd})}P_a$ only.

Applying the transformations (3.1.5) to our results (2.7.3), (2.7.4), and (2.7.5) and writing them in terms of the labels d, J, and n defined above we have

$$P_a^{\mathrm{I}}(r, s; p) = -(-1)^{a(k+1)+k(k+1)/2} R_a H(d, 2s; |p|^{1/(2s(r-2s))}) \times (\theta_2(0, -|p|^{r/(r-2s)})\, \theta_\tau(\pi r d/2s, |p|^{r/s}))^{-1} \tag{3.1.18a}$$

$$P_a^{\mathrm{II}}(r, s; p) = P_a^{\mathrm{III}}(r, r-s; p) \tag{3.1.18b}$$

$$P_a^{\mathrm{III}}(r, s; p) = R_a H(d, r-s; p^{1/(4s(r-s))}) \times (\theta_4(\pi/4, p^{r/4s})\, \theta_1(\pi s d/(r-s), p^{r/(r-s)}))^{-1} \tag{3.1.18c}$$

$${}^{(\mathrm{odd})}P_a^{\mathrm{III}}(r, r-1; p) = \lambda_{a,(a+J-n-1)/2}\, \tilde{\theta}_1(\pi a/r, p)/Q(p^{r/(r-2)}) \tag{3.1.18d}$$

$$P_a^{\mathrm{IV}}(r, s; p) = R_a H(d, r-2s; |p|^{1/(2s(r-2s))}) \times [\theta_4(\pi/4, |p|^{r/2s})\,\theta_1(\pi sd/(r-2s), -|p|^{r/(r-2s)})]^{-1} \tag{3.1.18e}$$

$$^{(\mathrm{odd})}P_a^{\mathrm{IV}}(r, (r-1)/2; p) = \gamma_{a,(a+J-n-1)/2}\,\theta_{\tau'}(\pi a/2r, p)/Q(-|p|^{r/(r-2)}) \tag{3.1.18f}$$

where

$$R_a = \theta_1(\pi as/r, p)/r \tag{3.1.19}$$

$$H(d, v; q) = \theta_3\left[\frac{\pi}{2}\left(\frac{d}{v} - \frac{a}{r}\right); q\right] - \theta_3\left[\frac{\pi}{2}\left(\frac{d}{v} + \frac{a}{r}\right); q\right] \tag{3.1.20}$$

In (3.1.18a) $\tau = 1$ if d is even and 4 if d is odd, and in (3.1.18f) $\tau' = 1$ if a is even and 2 if a is odd. In (3.1.18c) $s \neq r-1$, and in (3.1.18f) $s \neq (r-1)/2$ if r is odd. Further we recall from (2.7.1) and (2.7.2) that p is positive in Regimes II and III and negative in Regimes I and IV.

When $s = 1$ in the above expression ($s = r - 1$ in Regime III) we showed in A that at $p = 0$

$$P_a = \frac{4}{r}\sin^2\frac{\pi a}{r} \tag{3.1.21}$$

for each regime and further gave the next term in the expansion of the P_a as a function of p [Eqs. (3.3.22) of A]. We can calculate similar expansions for $s \neq 1$ from the above results. For Regime IV with $s = (r-1)/2$, r odd, we first need to note from (3.1.12) and (3.1.13) that for $0 \leqslant m \leqslant (r-3)/4$

$$g_{2m,a} \sim \begin{cases} 2(-1)^{(r-3)/2} e^{-\pi ia(r-4m-1)/4r} \sin\dfrac{\pi a}{4r}(r-4m-1), & a \text{ even} \\ i(e^{-i\pi a(r-4m-1)/2r} - (-1)^{(r-1)/2}), & a \text{ odd} \end{cases} \tag{3.1.22}$$

while for $0 \leqslant m \leqslant 3(r-3)/4$ and $(r-3)/2$ even

$$g_{(1/2)(r-3)+2m,a} \sim \begin{cases} 2|p|^{2m/(r-2)} e^{-\pi i(1-2m)/2r} \sin\dfrac{\pi a}{2r}(2m+1), & a \text{ even} \\ 2i|p|^{2m/(r-2)} e^{-\pi i(1-2m)/2r} \cos\dfrac{\pi a}{2r}(2m+), & a \text{ odd} \end{cases} \tag{3.1.23}$$

and for $0 \leqslant m \leqslant 3(r-5)/4 + 1$, $(r-3)/2$ odd, a even or odd

$$g_{(1/2)(r-1)+2m,a} \sim 2|p|^{(2m+1)/(r-2)} e^{\pi iam/r} \sin\frac{\pi a}{r}(m+1) \tag{3.1.24}$$

We find in each of the cases $s > 1$ ($s \neq r-1$ in Regime III), that with one exception to be noted below:

$$P_a(r, s; p) \sim \sum_{n=1}^{x} C_{a,n} p^{-b_n} + \frac{4}{r} \sin^2 \frac{\pi a s}{r} \tag{3.1.25}$$

Here x and the $C_{a,n}$ may depend on r, s, the phase and regime. Also $x \geqslant 1$ and the $C_{a,n}$ have the property

$$\sum_{a=1}^{r-1}{}^{*} C_{a,n} = 0, \qquad 1 \leqslant n \leqslant x \tag{3.1.26}$$

[the asterisk has the same meaning as in (1.6.9)], and the b_n are positive. In Regime I the largest b_n is

$$b_{\max} = \frac{s^2 - 1}{4s(r-2s)}, \quad \text{d odd}; \qquad b_{\max} = \frac{(s-r)^2 + s^2 - 2}{8s(r-2s)}, \quad \text{d even} \tag{3.1.27}$$

while in Regime II, Regime III $[s \neq (r-1)]$ and Regime IV $[s \neq (r-1)/2$, r odd$]$ we have $b_{\max}$ given by

$$\frac{(r-s)^2 - 1}{8s(r-s)}, \qquad \frac{s^2 - 1}{8s(r-s)}, \qquad \frac{s^2 - 1}{4s(r-2s)} \tag{3.1.28}$$

respectively. In Regime IV with $s = (r-1)/2$, r odd we have

$$b_{\max} = \frac{3(r-3)(r-5)}{8(r-2)^2} \tag{3.1.29}$$

Hence in these case the P_a diverge at criticality. This is possible, since from Section 1.4 we know these case are unphysical (i.e., there are negative Boltzmann factors), so the P_a do not have to be positive. The normalization condition (1.2.8) implies these divergent terms must cancel when summed over a, which is what we observe in (3.1.26).

The expansion (3.1.25) has the remarkable feature of containing as the constant term (i.e., term independent of p)

$$\frac{4}{r} \sin^2 \frac{\pi s a}{r} \tag{3.1.30}$$

which is the obvious generalization of the critical density (3.1.21) in the physical case $s = 1$.

Since the P_a are diverging at criticality, it is not possible to define exponents of the order parameters in the usual sense. However, as commented above, there is one nonphysical case in which the P_a do not diverge. This is in Regime IV with $r = 5$ and $s = 2$. From (3.1.15), (3.1.18f), (3.1.22), and (3.1.24) we deduce the expansions

$$^{(\mathrm{odd})}P_a^{\mathrm{IV}}(5, 2; p) \sim \frac{4}{5}\sin^2\frac{2\pi a}{5} - \frac{8}{5}(-1)^{a/2}\left(\sin\frac{\pi a}{10}\right)\left(\sin\frac{\pi a}{5}\right) \times \left[\cos\pi\frac{(J-n)}{3}\right] p^{1/9} \tag{3.1.31a}$$

$$^{(\mathrm{odd})}P_a^{\mathrm{IV}}(5, 2; p) \sim \frac{4}{5}\sin^2\frac{2\pi a}{5} - \frac{8}{5}(-1)^{(a+1)/2}\left(\sin\frac{\pi a}{5}\right)\left(\cos\frac{\pi a}{10}\right) \times \left[\sin\pi\frac{(J-n)}{3}\right] p^{1/9} \tag{3.1.31b}$$

for a even and a odd, respectively. Thus the critical exponent $\beta^{(5)}$ in this case is 1/9.

3.2. Free Energy

The free energy of the SOS model defined by the weights (1.2.2) can be calculated using the inversion relation method.[6] From the symmetries of the partition function given in Section 1.4 it suffices to calculate the free energy of the regions given in Fig. 5. Indeed if we consider the model as defined in terms of the conjugate modulus variables by (1.4.2) it suffices to calculate the free energy in Regime III_G ($\mu \neq r-1$), Regime II_G as given by (1.6.10) and the regime

$$|y/x^2|^{1/2} < |w| < |1/yx^2|^{1/2}, \qquad y = x^r \tag{3.2.1}$$

which we will denote Regime VIII_G. To see first define ρ' in (1.2.2) by

$$\rho' = \rho_0[h(2\eta)]^{-1}\exp[\pi(v^2-\eta^2)/2KK'] \tag{3.2.2a}$$

$$\rho' = \rho_0[h(2\eta)]^{-1}\exp[\pi(v^2-\eta^2)/2KL'] \tag{3.2.3b}$$

for $p > 0$ and $p \lesssim 0$, respectively. Then one can check the symmetry properties of the weight function W (which we will consider as a function of w, x, and y as defined by (1.4.4), (1.4.5), and (1.4.6)

$$W(w, x, y) = w^{1/2}W(1/w, y/x, y) \tag{3.2.3a}$$

$$= w(y/x)^{1/2}\, W(wy, x, y) \tag{3.2.3b}$$

where the = sign means the partition function given by the equated weights are the same [thus the actual weights may differ by a factor of the form

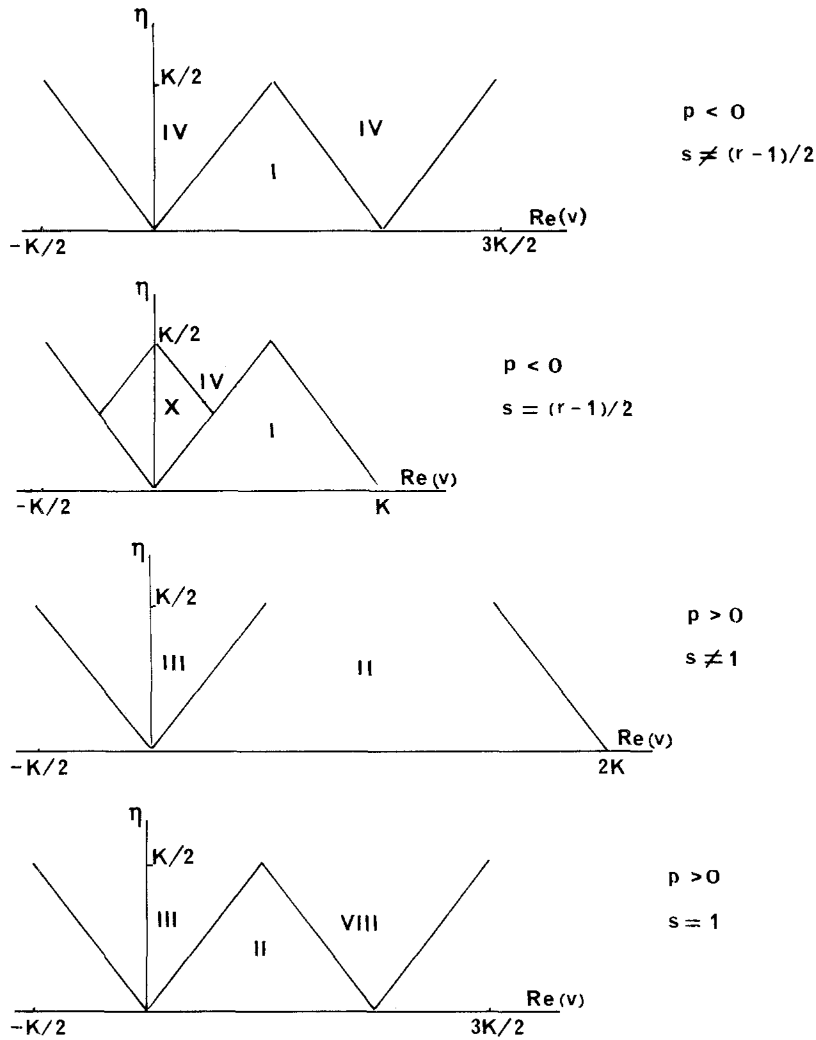


Fig. 5. Regimes I to X in the domain $0 < \eta < K/2$, $-\eta < \mathrm{Re}(v) < 2K - \eta$. The P_a and partition function Z in the portion of the $\mathrm{Re}(v) - \eta$ plane not shown above can be deduced from the symmetry relations given in Section 1.4.

(1.3.1)]. The symmetry (3.2.3a) maps Regime III_G to Regimes I and II; Regime II_G to Regime III with $s = r - 1$ and to Regime IV with $s = (r - 1)/2$; Regime VIII_G to Regime X. The symmetry (3.2.3b), after one iteration, maps Regime III_G to Regime IV′ (we consider this regime explicitly since at $p = 0$ it coexists with Regime VIII).

Let us denote $\kappa^T = \kappa^T(w, x, y)$, where T denotes the regime, κ the free energy per site, and w, x, y are given by (1.4.4) and (1.4.5) for Regimes II, III, and VIII and by (1.4.4) and (1.4.6) for Regimes I, IV, and X. Further denote $R\kappa^T = w^{1/2}\kappa^T(1/w, y/x, y)$. Then from the discussion above we have the relations

$$
\begin{aligned}
&\kappa^{\mathrm{I}} = R\kappa^{\mathrm{III}_G} && \kappa^{\mathrm{II}} = R\kappa^{\mathrm{III}} \\
&\kappa^{\mathrm{III}} = \kappa^{\mathrm{III}_G}(y^{r-1} \neq x^r) && \kappa^{\mathrm{III}} = R\kappa^{\mathrm{II}_G}(y^{r-1} = x^r) \\
&\kappa^{\mathrm{IV}} = \kappa^{\mathrm{III}_G}(y^{(r-1)/2} \neq x^r) && \kappa^{\mathrm{IV}} = R\kappa^{\mathrm{II}_G}(y^{(r-1)/2} = x^r) \\
&\kappa^{\mathrm{VIII}} = \kappa^{\mathrm{VIII}_G} && \kappa^{\mathrm{X}} = R\kappa^{\mathrm{VIII}_G} \\
&\kappa^{\mathrm{IV}'} = (w^2y^2/x)\,\kappa^{\mathrm{III}_G}(wy^2, x, y)
\end{aligned}
\tag{3.2.4}
$$

With ρ' given by (3.2.2), let us further define the parameters λ, u, and q by

$$
\lambda = 2\pi\eta/K', \qquad u = \pi(\eta + v)/K', \qquad q = e^{-2\pi K/K'} \tag{3.2.5a}
$$

for $p > 0$, while for $p < 0$ define

$$
\lambda = \pi\eta/L', \qquad u = \pi(\eta + v)/2L', \qquad q^2 = -e^{-\pi K/L'} \tag{3.2.5b}
$$

Then the definition of the weights (1.2.2) coincides with that of the weights given by Eqs. (6.1)–(6.5) of Ref. 6. The free energy in Regimes III_G, II_G, and VIII_G can then be written down immediately from equations (6.32)a, b, c, respectively, of Ref. 6 (although only the case of q^2 positive was explicitly considered in these equations, this restriction is not necessary). Define the functions Y by

$$
Y_1(w, x, y) = \sum_{n=1}^{\infty} \frac{[1-(x/w)^n](1-w^n)(x^n + x^{-n}y^n)}{n(1-y^n)(1+x^n)} \tag{3.2.6a}
$$

$$
Y_2(w, x, y) = \sum_{n=1}^{\infty} \frac{(1-w^{-n})\{(x^n-y^n)[(x/y)^n-(w/x^2)^n]+(1-x^n)(w^n-x^{-n})\}}{n(1-y^n)((x/y)^n+x^{-n})} \tag{3.2.6b}
$$

$$
Y_3(w, x, y) = \sum_{n=1}^{\infty} \frac{(1+x^n)[x^n+(y/x)^n]-(x^{-n}+x^n)[(wy)^n+(x/w)^n]}{n(1-y^n)(1+x^n)} \tag{3.2.6c}
$$

Then we have

$$
\log \kappa^{\mathrm{III}_G}(w, x, y) = \log \rho_0 + Y_1(w, x, y) \tag{3.2.7}
$$

$$\log \kappa^{\mathrm{II}_G}(w, x, y) = \log \rho_0 + \frac{\log w \log(x^3/y)}{2 \log(x^2/y)} + Y_2(w, x, y) \qquad (3.2.8)$$

$$\log \kappa^{\mathrm{VIII}_G}(w, x, y) = \log \rho_0 + \frac{1}{4} \log w^2 x + Y_3(w, x, y) \qquad (3.2.9)$$

We are particularly interested in the leading order singular behavior near criticality. To do this we apply the Poisson summation formula[5] to express the sums (3.2.6) as a series of Fourier integrals. The leading order behavior near criticality is then deduced from the $n = 1$ term of the series, by deforming the contour of integration around the poles of the integrand in the upper half-plane. Hence we deduce the leading order singular term in each of the regimes in Fig. 5 to be proportional to

Regime I:	$(-p)^{r/(r-2s)}$	(3.2.10a)
Regime II:	$p^{r/(r-2)}$ when $s = 1$; $p^{r/2(r-s)}$ otherwise	(3.2.10b)
Regime III:	0 when $s = 1$, r odd; $p^{r/2} \log p$ when $s = 1$, r even	
	$p^{r/2s}$ otherwise	(3.2.10c)
Regimes IV and IV′:	$(-p)^{r/2s}$	(3.2.10d)
Regime VIII:	$p^{r/2}$	(3.2.10e)
Regime X:	$(-p)^r \log \lvert p \rvert$	(3.2.10f)

where by 0 we man $\log \kappa$ is in fact an analytic function of p at $p = 0$. In (3.10a, b, d, e) when r and s are such that the exponent of p is an integer the term is to be multiplied by $\log \lvert p \rvert$.

Note that since we are only considering the portion of the $\mathrm{Re}(v) - \eta$ plane given in Fig. 5 we have $0 < s/r < 1/2$ (r, s relatively prime). When $s = 1$ in Regimes I–IV these values agree with those given in A.

4. SUMMARY

We have completed a study of the restricted SOS model with weights given by (1.2.2) or equivalently (1.4.2), began in an earlier paper[1] (referred to throughout as A). Whereas in A we considered only the case $\eta = K/r$, r a positive integer $\geqslant 4$, here we considered all cases $\eta = sK/r$, s and r relatively prime integers ($r \geqslant 4$).

Both the free energy and local height probabilities are calculated. However in all cases except $s = 1$ the model is unphysical: there are negative Boltzmann factors. Thus the local height probabilities can be negative. In fact we find that at criticality they diverge. The singular part of the free energy does not diverge at criticality. We given the leading order behavior in both cases.

The calculation of the local height probabilities is of independent mathematical interest. Using the corner transfer matrix technique we obtain expressions for this quantity in terms of m-fold sums. We express these sums (which are generalizations of combinatorial sums in Schur's[4] proof of the Roger's–Ramanujan identities) in terms of Gaussian polynomials, and show that in the limit $m \to \infty$ they are modular forms. Thus the study of the statistical mechanical model has resulted in the discovery of further generalizations of the Rogers–Ramanujan identities (which are presumably related to Gordon's generalization[7,9]).

APPENDIX

The corner transfer matrix (C.T.M.) technique when applied to the definition (1.2.7) of the P_a has been described in detail in Appendix A of paper A. In particular we were able to obtain tractable expressions for the P_a [Eq. (A26) of paper A] in domains analogous to Regime III_G with $\mu \neq r-1$ (1.4.10) and Regime II_G (1.6.10). The final expressions, analogous to (1.6.3), were then obtained by studying a special limiting case.

The only difference needed in the present case to the procedure given in paper A to derive (A26) is essentially one of notation. Here we work with the conjugate modulus form of the weights (1.4.2), where the variable is w, rather than the weights (1.2.2), where v is the variable. Other than this Eqs. (A1)–(A25) represent the necessary working to obtain tractable expressions for the P_a in Regime III_G ($\mu \neq r-1$) and Regime II_G. Thus following these workings step by step we derive the expression for P_a,

$$P_a = \text{Tr}\, S_a R_1^2 e^{-2\pi t\eta\mathscr{H}/M'} / \text{Tr}\, R_1^2 e^{-2\pi t\eta\mathscr{H}/M'} \tag{1}$$

where

$$(S_a)_{\mathbf{l}\mathbf{l}'} = \delta(l_1, a)\, \delta(\mathbf{l}, \mathbf{l}') \tag{2}$$

$\exp[\pi(v-\eta)\mathscr{H}/M']$ is the diagonal form of the single C.T.M. A, and in Regime III_G $\mu \neq r-1$,

$$t = 2, \qquad (R_1)^2_{\mathbf{l}\mathbf{l}'} = E(x^{l_1}, y)\, \delta(\mathbf{l}, \mathbf{l}') \tag{3}$$

while in Regime II_G

$$t = 2 - r, \qquad (R_1)^2_{\mathbf{l}\mathbf{l}'} = x^{l_1^2/4} y^{-l_1/4} E(x^{l_1}, y)\, \delta(\mathbf{l}, \mathbf{l}') \tag{4}$$

[we have used the notation $\delta(\mathbf{l}, \mathbf{l}') = \prod_{k=1}^{m} \delta(l_k, l_k')$].

Recall from (1.4.4) that

$$x^{1/2} = e^{-2\pi\eta/M'} \qquad \text{and} \qquad w^{1/2} = e^{\pi(v-\eta)/M'} \tag{5}$$

which expresses the above results in terms of x and w. In (2), (3), and (4) $\mathbf{l} = (l_1, l_2, ..., l_m)$ denotes the first m heights of the center row, beginning at the center height and moving right (recall from Section 1.6 the heights l_{m+1}, l_{m+2} and the fixed boundary heights of the center row). Further note the definition of the diagonal matrix $\mathscr{H}$ used here differs from that used in paper A by the scalar factor π/M'.

The matrix A is dependent on the boundary conditions. It is defined in terms of the local face transfer matrices[5] U_j where

$$(U_j)_{\mathbf{l}\mathbf{l}'} = W(l_j, l_{j+1} | l_{j-1}, l'_j) \prod_{\substack{k=1 \\ \neq j}}^{m} \delta(l_k, l'_k) \tag{6}$$

We have

$$A = F_2 F_3 \cdots F_{m+1} \tag{7}$$

where

$$F_j = U^{(j)}_{m+1} U^{(j)}_m U_{m-1} \cdots U_j \tag{8}$$

the superfixes on U_{m+1} and U_m denoting the fact that these matrices (but not $U_2, ..., U_{m-1}$) depend on the boundary heights and hence the value of j in (8).

The weights (1.4.2) are analytic functions of $w^{1/2}$, and thus from (5) are analytic functions of v of period $2iM'$. Hence within both Regimes II_G and III_G the elements of the C.T.M. A and its diagonal form $\exp[(v-\eta)\pi\mathscr{H}/M']$ are analytic functions of v of period $2iM'$ and thus of the form

$$[e^{(v-\eta)\pi\mathscr{H}/M'}]_{\mathbf{l}\mathbf{l}'} = w^{N(\mathbf{l})/2}\, \delta(\mathbf{l}, \mathbf{l}') \tag{9}$$

Assuming that $\mathscr{H}$ does not change discontinuously with x, the integers $N(\mathbf{l})$ must be independent of x. They can therefore be obtained by studying a particular case, the obvious choice being one in which the C.T.M. A is diagonal (or near diagonal). Note from (6) the off-diagonal elements of U_j (and thus the C.T.M. A) involve the weights β_l, while the diagonal elements involve the weights γ_l and δ_l. Thus if we can choose a limit such that

$$|\beta_l^2/(\gamma_l \delta_l)| \to 0 \tag{10}$$

for $l = 2, 3, ..., r-2$ the C.T.M. A is in its diagonal form. If μ as defined by (1.4.7) is such that μ and r are relatively prime (as is the case with the potion of Regime III_G corresponding to Regimes II and III, and in Regime II_G where $\mu = 1$) then (10) is realized in the limit $x \to 0$, $w \simeq 1$. We find the weights (1.4.2) (setting $v = 1$) become

$$W(l, m' \mid l', m) = w^{c(l', l, m')}\, \delta_{l,m} \tag{11}$$

where the function c is defined by (1.6.2). The C.T.M. A is then diagonal with entries

$$\begin{aligned} A_{\mathbf{l}\mathbf{l}'} &= \prod_{j=1}^{m} [W(l_{j+1}, l_{j+2} \mid l_j, l_{j+1})]^j\, \delta(l, l') \\ &= w^{\phi(\mathbf{l})} \end{aligned} \tag{12}$$

where $\phi(\mathbf{l})$ is given by (1.6.5).

In the portion of Regime III_G corresponding to Regimes I and IV with r even, we only require $\mu/2$ and r to be relatively prime, so that μ and r have the common factor 2. In these cases (10) fails for $l = r/2$. This means nonzero off-diagonal elements of the C.T.M. A occur when

$$l_j \neq l'_j \qquad \text{and} \qquad l_{j-1} = l_{j+1} = l'_{j-1} = l'_{j+1} = r/2 \tag{13}$$

Thus A is block-diagonal, so can be readily be diagonalized: the diagonal form is again given by (12).

Hence in all cases we have

$$N(\mathbf{l}) = 2\phi(\mathbf{l}) \tag{14}$$

Replacing $\eta - v$ by $2t\eta$ in (9) and (5) gives

$$[e^{-2t\eta\pi\mathscr{K}/M'}]_{\mathbf{l}\mathbf{l}'} = x^{t\phi(\mathbf{l})}\, \delta(\mathbf{l}, \mathbf{l}') \tag{15}$$

Substituting (15) in (1) we obtain the results (1.6.4)–(1.6.6) and (1.6.12)–(1.6.14).

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